Quiz: Infinite Series and Separable Differential Equations

Fundamentals of Calculus II

Name: senior/seniorita correcto/a

No justification necessary

(Yes or No)

1. Yes Does 3/10 + 3/100 + 3/1000+ ... = 1/3?
2. Yes Is $\frac{dy}{dx} = 2xy$ a separable differential equation?
3. No Every series converges.

4. Express $7\pi + 7\pi^2 + 7\pi^3 + 7\pi^4 + \dots$ in sum notation. Does the series converge?
   This is a geometric series with a common ratio of π, written as
   $$\sum_{n=1}^{\infty} 7\pi^n$$
   Since, $\pi < 1$, the series diverges.
5. Solve $\frac{dy}{dx} + 3x^2 = 2x$
   This is a separable differential equation. Therefore, we can separate the variables and integrate:
   $$\int 1dy = \int -3x^2 + 2xdx \implies y = -x^3 + x^2 + c.$$
6. Solve $\frac{dy}{dx} + 3x^2 = 2x$ with $y(0) = 5$.
   We evaluate the solution above at $x = 0$ to get $c = 5$. Therefore,
   $$y = -x^3 + x^2 + 5$$

7. Does $4 + 4/3 + 4/9 + 4/27 + \ldots$ converge? If so, to what value?
   This again is a geometric series with $r = 1/3$. Therefore, the series converges to $\dfrac{4}{1-1/3} = 2 * 3 = 6$.

8. Find the value of $\sum_{n=0}^{5} \pi(n5^e)$. Be sure to simplify your answer.
   Write out the terms:
   $$0 + \pi 5^e + 2\pi 5^e + 3\pi 5^e + 4\pi 5^e + 5\pi 5^e = 15 * (\pi 5^e)$$

State and justify your thought process.

A zombie was spotted on UVM's campus. Panic is in the air. We need to figure out how dangerous the situation could become. Here's what we know:

- Zombie infections depend on the number of zombies
- Each zombie can infect 3 people per day.
- Luckly, only a single zombie is on UVM's campus today.

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9. Express the spread of zombies as a diffential equation.
   Letting Z be the number of zombies on campus and t the number of days from today, we have
   $$\frac{dz}{dt} = 3z$$
   since each zombie contributes three more zombies per day.

10. Find an equation for the number of zombies on campus by solving the differential equation.
    We can solve the differential equations by separating the variables,
    $$\int 1/z dz = \int 3dt \implies ln|z| = 3t + C$$
    So,
    $$z = Ce^{3t}$$
    Evaluating the expression at time $t = 0$, we have $c = 0$. Thus, $z = e^{3t}$.

11. Determine the number of zombies on UVM's campus after 5 days.
Five days corresponds to $t = 5$, so

$$z = e^{3*5} = e^{15}.$$

12. With your mathematical prowess, describe the type of spreading you expect to see. How quickly will the zombies to spread?
The model we derived above indicates exponential growth. Even after just 5 days, the number of zombies is huge: e^{15}.

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13. What is the form of a geometric series? Under what conditions does a geometric series converge?
A geometric series is of the form $\sum_{n=0}^{\infty} ar^n$, which converges for $|r| < 1$.

14. Bonus: derive the formula for the sum of an infinite geometric series.
A geometric series up to n is of the form: $a + ar + ar^2 + ar^3 + \cdots + ar^n$ (let's call this S_n)
Multiplying the series by r we get: $ar + ar^2 + ar^3 + \cdots + ar^{n+1}$ (we'll call this rS_n)
Then,

$$S_n - rS_n = a - ar^{n+1}$$

implying

$$S_n(1-r) = a - ar^{n+1} \implies S_n = \frac{a(1-r^{n+1})}{1-r}$$

As $n \to \infty$, $(r^{n+1}) \to 0$ when $|r| < 1$.
Therefore, if $|r| < 1$, we have

$$S_n = \frac{a}{1-r}.$$